Test of the quantumness of atom-atom correlations in a bosonic gas

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Abstract

It is shown how the quantumness of atom-atom correlations in a trapped bosonic gas can be made observable. Application of continuous feedback control of the center of mass of the atomic cloud is shown to generate oscillations of the spatial extension of the cloud, whose amplitude can be directly used as a characterization of atom-atom correlations. Feedback parameters can be chosen such that the violation of a Schwarz inequality for atom-atom correlations can be tested at noise levels much higher than the standard quantum limit.

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In optics correlations between photons in a light field have been demonstrated using a Hanbury Brown Twiss setup [1]. There the incoming light beam is split by a semitransparent mirror and the two outputs are measured with photodetectors. By displacing detectors normal to the detected beams and by introducing a variable time delay, from the coincidences of detections the intensity-intensity correlation of the quantized light field,

$$C(1,2) = \langle {}^{\circ}_{\circ}\, \hat{I}(1)\, \hat{I}(2)\, {}^{\circ}_{\circ} \rangle. \tag{1}$$

is obtained, where 1 and 2 represent the space-time coordinates (x_1, t_1) and (x_2, t_2), respectively. Here $\hat{I} = \hat{E}^{(-)} \hat{E}^{(+)}$ with $\hat{E}^{(\pm)}$ being positive- and negative-frequency parts of the (here scalar) electric field of the light beam and ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ denotes time and normal-ordering.

Interpreting the expectation value in Eq. (1) as being based on a proper probability density, application of the Cauchy–Schwarz inequality leads to

$$C(1,2) \leq \sqrt{C(1,1)\, C(2,2)}. \tag{2}$$

This inequality is based on the assumption of a classical random process with correlations describing either complete randomness or bunching of photons [2]. However, quantum states of light exist that cannot be described in this way and violate the inequality (2). This has been used as a criterion for true quantumness — or non-classicality — of the incoming light beam, and has been experimentally observed as antibunched light [3, 4, 5, 6].

These concepts can also be applied to ultracold bosonic gases, where the electric field is replaced by the matter-field $\hat{\phi}(x,t)$. However, different from optical correlations measured by absorbing photodetectors, atom correlations will not occur in normal operator ordering. Nevertheless, an analogy between photonic and atomic correlations can be drawn. In fact, atom-atom correlations have been recently measured in a way analogous to the Hanbury Brown–Twiss optical setup [7, 8, 9]. Furthermore, there have been also approaches where the measurement of losses due to three-body recombinations has been used to infer on those correlations [10, 11, 12].

In these techniques either the gas has to be released from the trap or atomic losses play a major role. One might, however, manipulate or drive the system in such a way as to map the internal atom-atom correlations into easily accessible mesoscopic observables, that may be observed *in situ* without loosing atoms from the trap. This is the approach taken in this Letter: By applying a continuous feedback control of the atomic cloud's center of mass and by observing the cloud size, it is shown that atom-atom correlations can be detected and used to test for the quantumness of atom-atom correlations, in close analogy to antibunching in the case of photons.

Consider an ideal bosonic gas with N atoms of mass m, that is kept in a harmonic trap of frequency ω. A feedback loop is continuously applied to compensate for the motion of the center of mass of the gas $\hat{X}$, i.e. to damp the collective motion of atoms. It consists of the continuously repeated application of a measurement of $\hat{X}$ with measurement outcome $X_{\rm m}$ and resolution σ_0, and a corresponding shift by $-\zeta_0 X_{\rm m}$. Both processes shall be much faster than the free oscillation of the system with trap frequency ω, so that an instantaneous action can be assumed.

Experimental implementations of such a feedback loop are based on the collective interaction of the atoms with far off-resonant optical probe fields. Two-photon transitions, whose strength depend on the positions of atoms in the probe field, then lead to a redistribution of intensities in optical-field modes. The latter may be detected and used as an input signal for a subsequent control action using optical phase shifts of the same probe field. Such a scheme has been realized with atoms in optical lattices [13] or may be realized as an extension of the single-atom experiment by Fischer *et al.* [14]. Possibly it may also be integrated into the magneto-optical or optical trap configuration, by detecting and modulating the trapping laser fields.

Given the continuous application of feedback at rate γ, two parameters determine the feedback dynamics: the rms time-integrated measurement resolution $\sigma = \sigma_0/\sqrt{\gamma}$ and the feedback shift rate $\zeta = \zeta_0\gamma$. The time evolution of the N-atom density operator of the system, $\hat{\varrho}_N$, is then described by the master equation of quantum Brownian motion [15, 16, 17],

$$\partial_t\, \hat{\varrho}_N = -\frac{i}{\hbar}[\hat{H}, \hat{\varrho}_N] + i\frac{\zeta}{2\hbar}[\hat{P}, \{\hat{X}, \hat{\varrho}_N\}] - \frac{1}{8\sigma^2}[\hat{X}, [\hat{X}, \hat{\varrho}_N]] - \frac{\zeta^2\sigma^2}{2\hbar^2}[\hat{P}, [\hat{P}, \hat{\varrho}_N]]. \tag{3}$$

Here $\hat{H}$ is the Hamiltonian of the non-interacting atomic gas in the trap potential and the center-of-mass (cm) operator and its canonically conjugate total momentum read

$$\hat{X} = \frac{1}{N}\int dx\, \hat{\phi}^\dagger(x)\, x\, \hat{\phi}(x), \qquad \hat{P} = -i\hbar \int dx\, \hat{\phi}^\dagger(x)\, \partial_x\, \hat{\phi}(x), \tag{4}$$

with the atomic field $\hat{\phi}(x)$ obeying the bosonic commutator relation $[\hat{\phi}(x), \hat{\phi}^\dagger(x')] = \delta(x - x')$.

The stationary behavior of the cm, obtained from Eq. (3), is an exponential damping at rate $\zeta/2$ of the coherent oscillation: $\lim_{t\to\infty}\langle \hat{X}(t)\rangle = 0$. Moreover, its rms spread converges exponentially at the same rate to the non-zero stationary value

$$\lim_{t\to\infty} \sqrt{\langle [\Delta\hat{X}(t)]^2\rangle} = \Delta X_{\rm s}, \tag{5}$$

where $\Delta\hat{X} = \hat{X} - \langle\hat{X}\rangle$. It represents the noise left in the cm after a time of the order of ζ^{-1} needed for damping the coherent cm oscillation. This noise is determined solely by the parameters of the

feedback and trap:

$$\Delta X_{\rm s} = \delta X_0 \sqrt{(\eta + \eta^{-1})/2}, \tag{6}$$

with the rms spread of the cm in the ground state of the trapping potential being

$$\delta X_0 = \sqrt{\hbar/(2Nm\omega)}, \tag{7}$$

which serves as the standard quantum limit (SQL) for the cm coordinate. The parameter

$$\eta = \delta {X_0}^2/(\zeta\sigma^2) \tag{8}$$

specifies the ratio of spatial localization due to the potential over that due to the feedback. Note, that Eq. (6) attains the SQL as a minimum value for $\eta = 1$ but is otherwise much larger.

Our goal is to describe atomic correlation effects in the dynamics of the atomic density. For that purpose we need the single-atom density matrix

$$\rho(x, x', t) = \langle \hat{\phi}^\dagger(x')\, \hat{\phi}(x) \rangle_t, \tag{9}$$

with $\langle \ldots \rangle_t = \mathrm{Tr}[\ldots \hat{\varrho}_N(t)]$. To obtain the dynamical evolution of this density matrix from Eq. (3) of course is prevented by the correlations in the many-atom systems. That correlations play a role in Eq. (3) can be seen from the occurrence of products of operators $\hat{X}$ and $\hat{P}$, that contain products of four field operators [cf. Eq. (4)], similar to atom-atom interactions. However, recently it has been shown [18], that despite of these problems, the single-atom density matrix can be obtained via a procedure that we may briefly outline here:

Instead of the single-atom density matrix, the joint Wigner function of single atom (variables x, p) and cm of the other $N-1$ atoms (variables X, P) is considered:

$$\begin{aligned} W(x, p; X, P, t) \;&=\; (2\pi\hbar)^{-3} \int dx' e^{-ix'p/\hbar} \int dX' \int dP' \\ &\times \left\langle \hat{\phi}^\dagger\!\Big(x - \frac{x'}{2}\Big)\, e^{i[(\hat{P}-P)X' + (\hat{X}-X)P']/\hbar} \hat{\phi}\Big(x + \frac{x'}{2}\Big) \right\rangle_t . \end{aligned} \tag{10}$$

From Eq. (3) a closed Fokker-Planck equation follows for this distribution, which is of linear type with positive semi-definite diffusion matrix, leading thus to a bound analytic Green function of Gaussian type [19]. In consequence, given the initial conditions, analytic solutions for this Wigner function can be obtained, from which the solution for the single-atom density matrix are derived by integration over the auxiliary phase-space variables:

$$\rho(x + x', x - x', t) = \int dX \int dP \int dp\, W(x, p; X, P, t)\, e^{2ipx'}. \tag{11}$$

Thus in principle the complete atomic density profile could be obtained. Here we focus on the rms spread of the corresponding atomic density,

$$\Delta x(t) = \left\{ \int \frac{dx}{N}\, x^2\, \rho(x,x,t) - \left[\int \frac{dx}{N}\, x\, \rho(x,x,t) \right]^2 \right\}^{\frac{1}{2}}, \tag{12}$$

giving us information on the quantum-statistically averaged temporal evolution of the extension of the atomic cloud. From a complete solution of the Fokker–Planck equation for (10) this variance can be shown to exponentially converge at rate $\zeta/2$ to the asymptotic behavior $\lim_{t\to\infty} \Delta x(t) \sim \Delta x_{\mathrm{a}}(t)$, defined by

$$\Delta x_{\mathrm{a}}(t) = \sqrt{\Delta X_{\mathrm{s}}{}^2 + \sigma_q{}^2(t)}. \tag{13}$$

In this equation the first term in the square root is given by the constant stationary rms cm spread [cf. Eq. (6)], whereas the second term is explicitely time dependent and reads

$$\sigma_q{}^2(t) = \int \frac{dx}{N} \left\langle \hat{\phi}^\dagger(x)\, [q(x,t)]^2\, \hat{\phi}(x) \right\rangle_0 - \int \frac{dx}{N} \int \frac{dx'}{N} \left\langle \hat{\phi}^\dagger(x)\, q(x,t)\, \hat{\phi}(x)\, \hat{\phi}^\dagger(x')\, q(x',t)\, \hat{\phi}(x') \right\rangle_0, \tag{14}$$

with the expectation value being taken with respect to the initial N-atom density operator $\hat{\varrho}_N(0)$. The explicit time dependence of Eq. (14) is given by the single-atom quadrature, defined as

$$q(x,t) = x\cos(\omega t) - \frac{i\hbar\partial_x}{m\omega}\sin(\omega t). \tag{15}$$

Equation (14) represents the central result of our Letter. In its second part it contains an atom-atom correlation function with four matter-field operators, showing that due to the feedback these correlations show up in the observable mesoscopic size of the cloud, cf. Eq. (13). Thus the atom-atom correlations will become visible in a purely single-atom property.

Due to its explicit time dependence, in general there will be no stationary size of the atomic cloud, but instead the cloud will periodically breath. It should be emphasized, that this breathing has nothing in common with the well-known collective oscillations of a Bose gas or condensate. The latter rely on the presence of atomic collisions, whereas the breathing discussed here is an effect solely produced by the feedback. Feedback of course effectively mediates interactions between atoms, so that a certain analogy to true atom-atom interactions can be drawn. However, whereas collisions of ultracold atoms lead to Hamiltonian terms, the feedback in addition provides non-unitary parts of the time evolution [cf. Eq. (3)].

Thus, after a transient behavior during a time of the order of $1/\zeta$, the cloud size will oscillate at twice the trap frequency, which resembles single-atom quadrature squeezing [20]. The amplitude

of this oscillation will of course depend on the initial quantum state and its correlations at time $t=0$, before the feedback has been turned on. Since the SQL for a single atom in the trap reads $\delta x_0 = \delta X_0 \sqrt{N}$, it follows that the atomic cloud size (13) can become smaller than that value. The condition for quadrature squeezing (QS) on the single-atom level would then be that at some time during the half period π/ω the following inequality is fulfilled:

$$\Delta x_{\mathrm{a}}(t) < \delta x_0 \qquad \text{(single-atom QS)}. \tag{16}$$

However, there is more to Eq. (13) than single-atom QS. It can be revealed by taking a closer look at the structure of Eq. (14) and applying the ideas developed in the context of photon anti-bunching. The probability density for finding an atom at position x is undoubtedly defined as

$$P(x) = \langle \hat{\phi}^\dagger(x)\hat{\phi}(x)\rangle / N. \tag{17}$$

A quasi joint probability density for two atoms being at positions x and x', that is consistent with the definition (17), reads

$$P(x,x') = \langle \hat{\phi}^\dagger(x)\hat{\phi}(x)\hat{\phi}^\dagger(x')\hat{\phi}(x')\rangle / N^2. \tag{18}$$

Note that different from the optical case this correlation is not normally ordered. Consistency means here that for N atoms the marginals of $P(x,x')$ reproduce the correct probability density:

$$\int dx'\, P(x,x') = P(x), \qquad \int dx\, P(x,x') = P(x'). \tag{19}$$

Clearly $P(x,x')$ is not a proper probability density in general. However, when interpreting the atomic fields as classical ones: $\hat{\phi}(x) \to \phi(x)$, it becomes a proper classical joint probability density. In this classical interpretation, we may now apply the Schwarz inequality that states

$$\int dx \int dx'\, q(x)q(x')P(x,x') \leq \int dx\, q^2(x)P(x) \qquad \text{(classically)}. \tag{20}$$

Applying this result to Eq. (14) one obtains a classical inequality for the contribution to the size of the atomic cloud: $\sigma_q^2(t) \geq 0$. In consequence, via Eq. (13) one arrives at the classical inequality for the size of the atomic cloud: $\Delta x_{\mathrm{a}}(t) \geq \Delta X_{\mathrm{s}}$.

A violation of the Schwarz inequality would indicate true quantum correlations between atoms as opposed to classical ones, since then the second expectation value in Eq. (14) cannot be described by integration over a proper probability density. The condition for this case is then

$$\Delta x_{\mathrm{a}}(t) < \Delta X_{\mathrm{s}} \qquad \text{(Schwarz violation)}. \tag{21}$$

Indeed, since Eq. (14) contains atom-atom correlations in the second term on the rhs, one may interpret a violation of the Cauchy–Schwarz inequality, as formulated in Eq. (21), as a test for true quantum correlations between atomic pairs. Note, that the concept of defining quantumness here is the same as in the case of photon antibunching in the context of the optical Hanbury Brown–Twiss experiment.

A Schwarz violation at time t can be easily shown to be equivalent to a violation of the (classical) inequality

$$\Delta q(t) \geq \sqrt{\langle [\Delta \hat{Q}(t)]^2 \rangle}, \tag{22}$$

where Δq and ΔQ are the rms spread of the ”single-atom” quadrature, corresponding to the atomic density, and the rms spread of the cm quadrature, respectively. The nature of such a violation can be understood by considering the special case where the Schwarz violation occurs at a time where the quadrature (15) reduces to the atomic position, i.e. $q(t) \rightarrow x$. The relation (22) then states, that classically the size of the atomic cloud Δx is always equal or larger than the rms spread of the cm of the cloud. In other words, the cm of the object is well localized within the spatial extension of the cloud. A violation of this classical inequality would then correspond to cases where the cm coordinate reveals a rms spread larger than the cloud's size. In the extreme case, the atomic cloud can then be seen as an almost pointlike object, its internal distribution not being resolved, whose (cm) coordinate fluctuates.

It is thus the transition from an atomic cloud with well localized cm to a quasi pointlike object with large fluctuation of its coordinate, that corresponds to a transition of classical to quantum atom-atom correlations in the cloud. Clearly for the general case, i.e. a Schwarz violation at an arbitary time withing the half period π/ω, the above interpretation correspondingly holds for a specific quadrature $q(t)$ instead of position.

In overall thus two levels of quantum or – if one wishes to use this term – non-classical behavior can be distinguished by the amplitude of the oscillation of the atomic-cloud size. These two however, single-atom QS and Schwarz violation, do not form a unique hierarchy: For different parameters η of the feedback mechanism the two boundaries [cf. Eqs (16) and (21)], appear in different orders. The order depends on whether $\Delta X_{\mathrm{s}} \leq \delta x_0$ or not. The parameter range for this condition is obtained as

$$\Delta X_{\mathrm{s}} \leq \delta x_0 \quad \text{for} \quad N - \sqrt{N^2 - 1} \leq \eta \leq N + \sqrt{N^2 - 1}. \tag{23}$$

In the case $N \rightarrow \infty$ this range includes all possible values of η and thus for a truly macroscopic

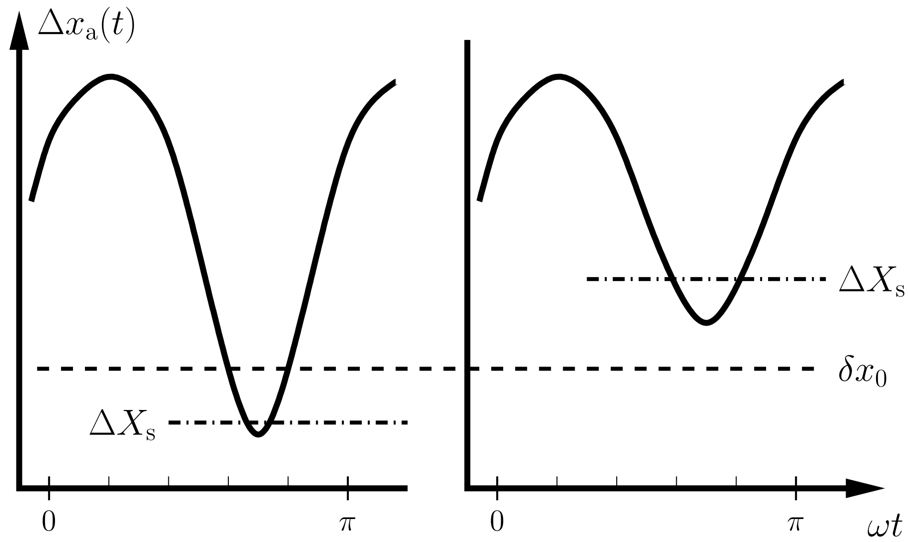

FIG. 1: Observable size of the cloud versus time: For the parameter range given by Eq. (23) (left part) the Schwarz boundary is below the one for QS whereas for values of η outside this range (right part) the reverse is true.

system, before a Schwarz violation can be observed always first single-atom QS appears. This case is depicted in the left part of Fig. 1.

However, for a finite system values for η can be found outside the range (23) which then reveals an exchange of the order of single-atom QS and Schwarz violation. In this case quantum correlations between atoms can be observed without the presence of single-atom QS, cf. right part of Fig. 1. The corresponding parameter range is given by

$$\eta < N - \sqrt{N^2 - 1} \quad \text{or} \quad \eta > N + \sqrt{N^2 - 1}. \tag{24}$$

In this range of η apparently the quantumness of correlations can be detected at much higher noise levels than the single-atom SQL δx_0, which may render its experimental observation substantially more feasible. Moreover, the two ranges in Eq. (24) correspond to weak and strong feedback localization, respectively, and thus may allow for a suitable combination of values for the shift rate ζ and the rms time-integrated measurement resolution σ in an experiment. Last but not least this scheme may be applied even for the extreme case of only two indistinguished bosonic atoms, for which the range of possible values of η, according to the case (24), becomes even more broad.

The criterium for quantumness of correlations can be tested directly from measurements of the size of the atomic cloud $\Delta x(t)$ over half a period of the trap oscillation, after the system has reached its asymptotic behavior within a delay time of the order of $1/\zeta$. Experimentally a suffi-

ciently large number of sequences of feedback evolutions of varying time duration and final cloud-size measurements have to be performed. Each sequence starts with the identically prepared initial quantum state, whose correlations are to be detected. Thus the final cloud-size measurements can be arbitrarily destructive and can be performed for example by density-profile measurements. These may be implemented by absorption imaging [21], dispersive light scattering [22], or possibly phase-modulation spectroscopy [23]. Thus together with possible experimental techniques to generate the required feedback loop [13, 14], an experimental implementation of the presented scheme and a test for quantum atom-atom correlations in bosonic gases seem to be feasible.

In summary we have shown that the quantumness of atom-atom correlations in a trapped bosonic gas can be made observable as size oscillations of the atomic cloud via feedback. For weak and strong feedback localization a Schwarz violation for atom-atom correlations of a gas with finite atom number can be observed in the absence of single-atom QS at correspondingly higher noise levels than the SQL. Together with the feasibility of implementing this scheme with present experimental techniques, this may allow for detecting the quantumness of atom-atom correlations in a bosonic gas.

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